Γ countable group

(X,μ) st. prob space $([0,1],\lambda)$

An action $\Gamma \curvearrowright X$

1) <u>measurable</u> $\underset{\text{Borel}}{A \subseteq X}$ $\gamma \cdot A$ is still Borel.

11) <u>probability meas. pres</u> (pmp) $\mu(\gamma A) = \mu A$

<u>Defn</u> Two free pmp $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$

are <u>Orbit equivalent</u> (OE)

if $f : X \to Y$ bijection, preserves measures.

plus, $f(\Gamma \cdot x) = \Lambda f(x)$

If Γ, Λ have OE free actions, Γ OE Λ

Q Which groups are OE?

Are $\mathbb{F}_2$ OE $\mathbb{F}_3$?

Rank $\to$ $\mathbb{F}_2 \not\cong \mathbb{F}_3$

Dye's thm $\mathbb{Z}$ OE $\mathbb{Z}^2$ OE $\mathbb{Z}^3$ ...

Def An eq. rel $E \subseteq X$.
countable Borel e.r (CBER)

1) E is Borel subset of X^2
2) It has countable eq classes

It is pmp if $\forall f: A \to B$ s.t.

$$f(x) E x \implies \mu A = \mu B$$

It is ergodic if $\forall \underset{\text{Borel}}{A \subseteq X}$ and E-inv.

$$\mu A = 0, 1$$

Ex $\underset{\text{free pmp}}{\Gamma \curvearrowright X}$, $x E_\Gamma y \iff \exists \gamma \in \Gamma, \gamma . x = y$

We say E, F isomorphic eq. if $f[x]_E = [f(x)]_F$
$\exists f: X \to Y$, bijection preserving measure

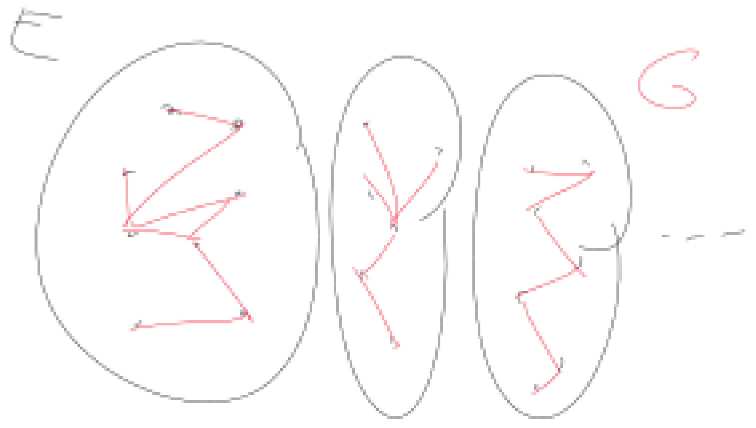
Ex $X = \left(2^{\Gamma}, (\text{fair coin})^{\otimes \Gamma}\right)$

$\Gamma \curvearrowright 2^{\Gamma}$ $\quad (\gamma \cdot x)(\delta) = x(\gamma^{-1}\delta)$

Bernoulli shift. Free a.e.

Defn A graphing for E

$\underset{\text{Borel}}{G \subseteq X^2}$ s.t. the conn comp are exactly the E-classes



Defn (!) [Levitt] $\text{Cost}(G) = \frac{1}{2}\int_X \deg(x)\, d\mu(x)$

$$\text{Cost}(E) = \inf_G \text{cost}(G)$$

Thm [Gaboriau] For any eq rel E and graphing G

$c(E) = c(G) \iff G$ is a tree on every component

Consequence $\operatorname{cost}\left(E_{\mathbb{F}_2}\right) = 2$

$$G = \left\{ (x, sx) : s \in S \right\}$$

$$\operatorname{cost}\left(E_{\mathbb{F}_3}\right) = 3$$

Thm [Hjorth] If E is a ergodic treeable and $\operatorname{cost} E = n$, or ∞, $E = E_{\mathbb{F}_n}$

2. Measure class pres. and type III

Def A CBER E is mcp

if $\forall f : A \to B$, $f(x) E x$

$\implies \mu A = 0 \iff \mu B = 0$

If mcp

It is type III if $\nexists\, \nu \sim \mu$ (σ-finite)

s.t. E preserves ν

($\nu \sim \mu \iff \forall A \subseteq X,\ \mu A = 0 \iff \nu A = 0$)

$\boxed{\mathbb{F}_2 \curvearrowright \partial \mathbb{F}_2}$

ergodic

Q If E is a treeable type III CBER

When $E = E_{\mathbb{F}_n}$? Is n unique?

Rem's Any treeable $E = \mathop{\ast}\limits_{i=0}^{\omega} E_i$ ← hyperfinite

Thm (P.)

If E is treeable non-amenable CBER

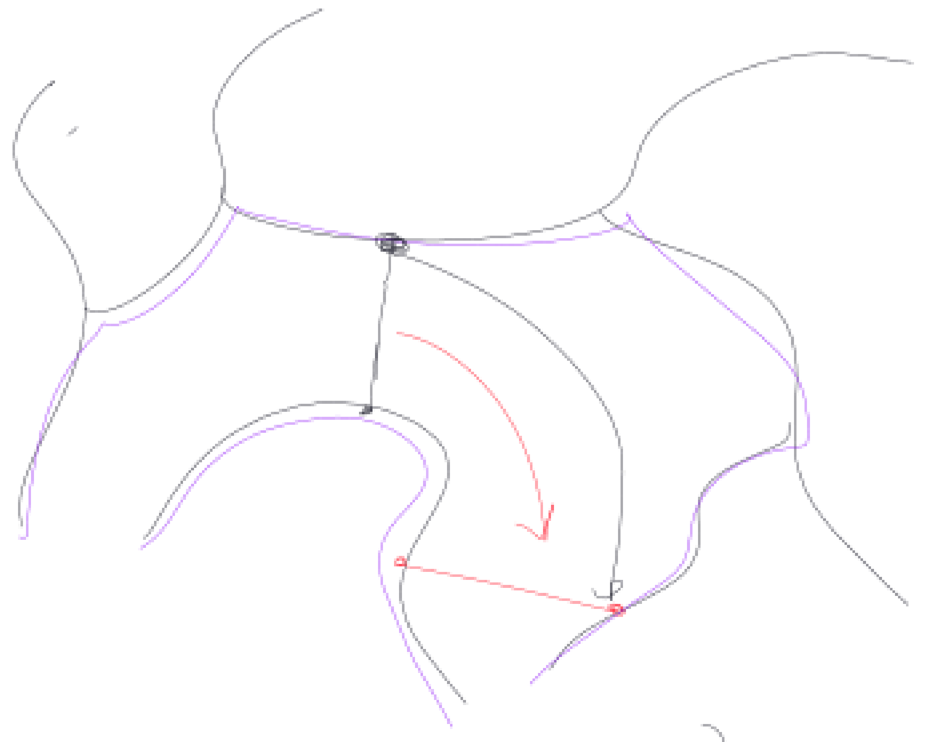
nowhere amenable

which is type III, $\forall n \in [2, \infty]$

$$E = E_{\mathbb{F}_n}$$

$\square$

~~$\Gamma = G *_H G'$ (non-trivial $[G:H]$, $[G':H] \geq 2,3$)~~



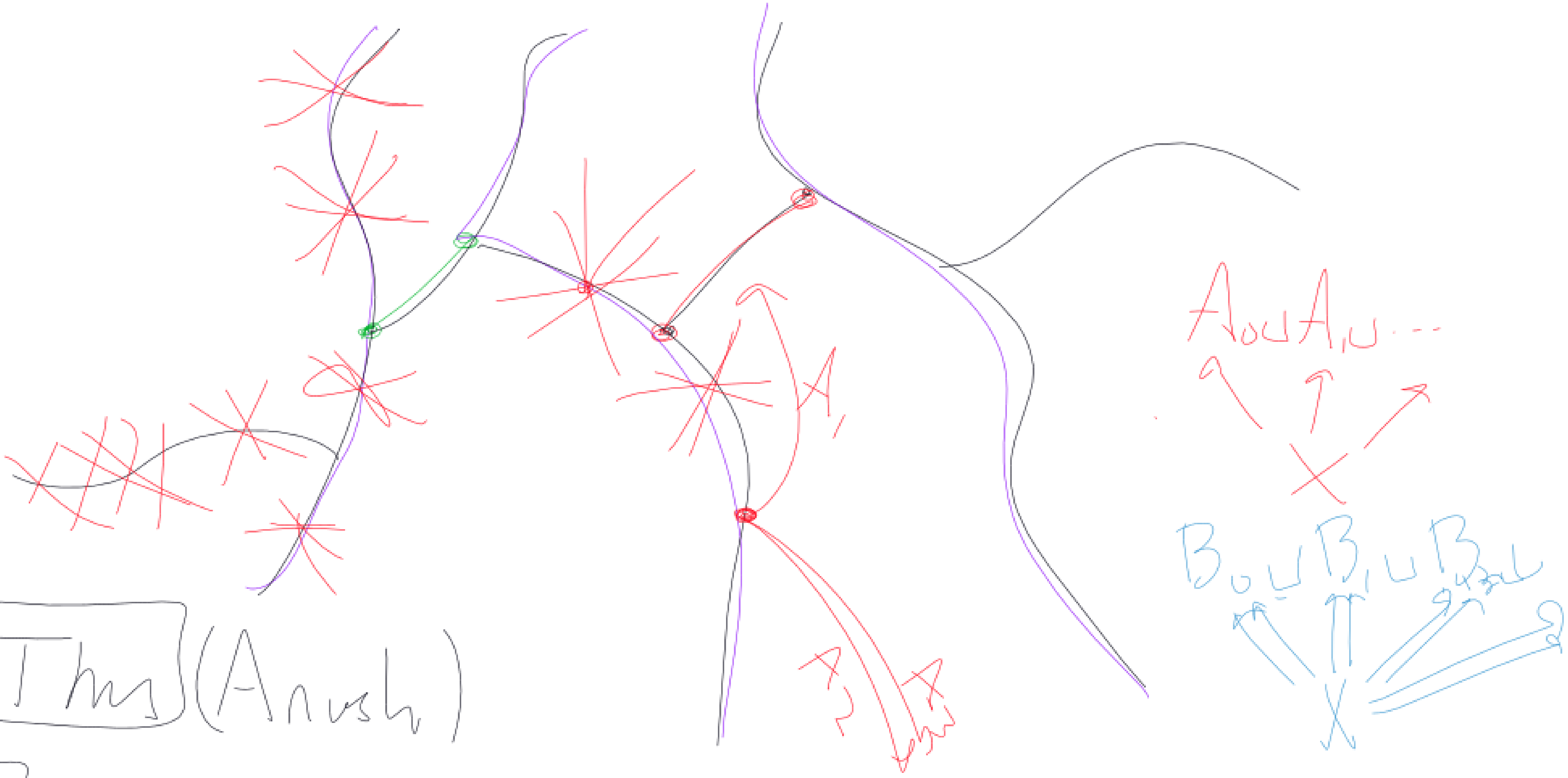
<u>Prop</u> If E is ergodic mg

$A, B \subseteq X$ non-null

1) If $\nu \sim \mu$ is preserved, $f : A \to B$ ' ' ' ' '

$\longleftrightarrow$ $\nu A = \nu B$

ii) If E is type III $\exists f$ always.



$\boxed{\text{Thm}}$ (Anush)

$\exists$ ~~amenable~~ ergodic subgraph + type III

$E = \mathcal{H}_1$ $\cdot_{12} = E_{\mathbb{F}_2}$

$$\begin{array}{c} \Gamma \curvearrowright X \\ \cup\!| \\ \mathbb{F}_2 \end{array}$$

$E_{\boxed{\mathbb{F}_2}}$ "smooth" in E_Γ